ON STRONGLY SUPERSOLUBLE FINITE GROUPS

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Throughout this report, all groups are finite. The notion of a normal subgroup takes a central place in the theory of groups. One of its generalizations is the notion of a modular subgroup, i.e. a modular element (in the sense of Kurosh [1, Chapter 2, p. 43]) of a lattice of all subgroups of a group. Recall that a subgroup M of a group G is called modular in G, if the following assertions hold:

1) $\langle X, M \cap Z\rangle = \langle X, M\rangle \cap Z$ for all $X \leq G, Z \leq G$ such that $X \leq Z$, and

2) $\langle M, Y \cap Z\rangle = \langle M, Y\rangle \cap Z$ for all $Y \leq G, Z \leq G$ such that $M \leq Z$.

Properties of modular subgroups were studied in the book [1]. Groups with all subgroups are modular were studied by R. Schmidt [1], [2] and I. Zimmermann [3]. By parity of reasoning with subnormal subgroup, in [3] the notion of a submodular subgroup was introduced.

Definition 1 [3]. A subgroup H of a group G is called submodular in G, if

there exists a chain of subgroups $H = H_0 \leq H_1 \leq \ldots \leq H_{s-1} \leq H_s = G$ such that H_{i-1} is a modular subgroup in H_i for $i = 1, \ldots, s$.

Using this notion we introduce a key notion of this report.

Definition 2. A group G we will call strongly supersoluble if G is supersoluble and every Sylow subgroup of G is submodular in G.

Denote $s\mathfrak{U}$ the class of all strongly supersoluble groups. The following results are obtained.

Theorem 1. *Let G be a group. Then the following hold:*

1) *if G is strongly supersoluble, then every subgroup of G is strongly supersoluble;*

2) *if G is strongly supersoluble and $N \trianglelefteq G$, then G/N is strongly supersoluble;*

3) *if $N_i \trianglelefteq G$ and G/N_i is strongly supersoluble for $i = 1, 2$, then $G/N_1 \cap N_2$ is strongly supersoluble;* 4) *if $H_i \trianglelefteq G$, H_i is strongly supersoluble, $i = 1, 2$ and $H_1 \cap H_2 = 1$, then $H_1 \times H_2$ is strongly supersoluble;*

5) *if $G/\Phi(G)$ is strongly supersoluble, then G is strongly supersoluble;*

6) *the class of groups $s\mathfrak{U}$ is a hereditary saturated formation.*

We denote $\mathfrak{V}$ the class of all abelian groups of exponent free from squares of primes.

Theorem 2. *The class of all strongly supersolubility groups is a local formation and has a local screen f such that $f(p) = \mathfrak{A}(p-1) \cap \mathfrak{V}$ for any prime p.*

Theorem 3. *Let the group $G = AB$ be the product of nilpotent subgroups A and B. If A and B are submodular in G, then G is strongly supersoluble.*

In Theorem 3 we can't discard the submodularity of one of subgroups.

Example. In group $G = AB$, where $A \simeq Z_{17}$ and $B \simeq Aut(Z_{17}) \simeq Z_{16}$, the subgroup A is submodular, but the subgroup B is not submodular in G. The group G is supersoluble, but not strongly supersoluble. The example also shows that $s\mathfrak{U} \neq \mathfrak{U}$.

Theorem 4. *A group G is strongly supersoluble if and only if G is metanilpotent and any Sylow subgroup of G is submodular in G.*

References

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